

# THE ORBIT METHOD FOR PROFINITE GROUPS AND A $p$ -ADIC ANALOGUE OF BROWN'S THEOREM

BY

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ABSTRACT

We develop an approach to the character theory of certain classes of finite and profinite groups based on the construction of a Lie algebra associated to such a group, but without making use of the notion of a polarization which is central to the classical orbit method. Instead, Kirillov's character formula becomes the fundamental object of study. Our results are then used to produce an alternate proof of the orbit method classification of complex irreducible representations of  $p$ -groups of nilpotence class  $< p$ , where  $p$  is a prime, and of continuous complex irreducible representations of uniformly powerful pro- $p$ -groups (with a certain modification for  $p = 2$ ). As a main application, we give a quick and transparent proof of the  $p$ -adic analogue of Brown's theorem, stating that for a nilpotent Lie group over  $\mathbb{Q}_p$  the Fell topology on the set of isomorphism classes of its irreducible representations coincides with the quotient topology on the set of its coadjoint orbits.

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## Introduction

The orbit method was originally discovered in the late 1950s – early 1960s by Alexandre Kirillov [Ki62] for connected and simply connected nilpotent Lie groups. If  $G$  is such a group and  $\mathfrak{g}$  is its Lie algebra, this method provides an explicit bijection between the **unitary dual**  $\widehat{G}$  of  $G$ , i.e., the set of equivalence classes of unitary irreducible representations of  $G$ , and the set  $\mathfrak{g}^*/G$  of orbits of the induced action of  $G$  on  $\mathfrak{g}^*$  (called **coadjoint orbits**). A major ingredient of this theory is **Kirillov’s character formula**. Roughly speaking, it states that if  $\Omega \subset \mathfrak{g}^*$  is a coadjoint orbit and  $\rho_\Omega \in \widehat{G}$  is the corresponding representation, then the character of  $\rho_\Omega$ , viewed as a generalized function on  $G$ , is the pullback via the logarithm map  $\log : G \rightarrow \mathfrak{g}$  of the inverse Fourier transform of a suitably normalized  $G$ -invariant measure on  $\mathfrak{g}^*$  supported on  $\Omega$ .

Since then Kirillov’s approach has been extended to many other classes of groups: nilpotent  $p$ -adic Lie groups [Mo65],  $p$ -groups of nilpotence class  $< p$  (beginning with [Ka77]), and uniformly powerful (or uniform, for short) pro- $p$ -groups [Ho77, JZ06], to name the ones that will appear in this paper<sup>1</sup>. Each such extension usually involves two modifications: one has to work with a correct analogue of a “unitary irreducible representation” in each context, and one has to find an appropriate version of the Lie algebra construction. For example, if  $G$  is a  $p$ -adic Lie group, its Lie algebra is defined as usual, but  $\widehat{G}$  has to be understood as the set of isomorphism classes of irreducible complex “algebraic” (or, in a different terminology, “smooth”) representations of  $G$ . On the other hand, if  $G$  is a  $p$ -group of nilpotence class  $< p$  (respectively, a uniform pro- $p$ -group), then  $\widehat{G}$  has to be understood as the set of isomorphism classes of continuous complex irreducible representations of  $G$ , and the usual Lie algebra construction is replaced by a construction of Lazard which produces a finite Lie ring [Khu98] (respectively, a uniform Lie algebra over  $\mathbb{Z}_p$  [DDMS]) associated to  $G$ .

After these modifications have been made, the theory follows the pattern of Kirillov’s original approach (modulo various technical difficulties). Namely, in each case the underlying additive group of  $\mathfrak{g}$  has a natural topology, and  $\mathfrak{g}^*$  can

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<sup>1</sup> The orbit method can also be applied, with suitable changes, to solvable Lie groups; moreover, its philosophy extends essentially to all Lie groups, and even beyond them. However, these generalizations lie in a different direction from the ones considered in this article.

be identified with the **Pontryagin dual** of  $\mathfrak{g}$ . Given an element  $f \in \mathfrak{g}^*$ , one looks for a **polarization** of  $\mathfrak{g}$  at  $f$ , i.e., a **Lie subalgebra**  $\mathfrak{h} \subseteq \mathfrak{g}$  which has the property that  $f$  is trivial on  $[\mathfrak{h}, \mathfrak{h}]$ , and which is maximal among all **additive subgroups** of  $\mathfrak{g}$  with this property. Polarizations always exist, and if  $H$  is the subgroup of  $G$  corresponding to a polarization  $\mathfrak{h}$ , then  $f$  induces a 1-dimensional character  $\chi_f$  of  $H$  and we can form the induced representation  $\rho_f = \text{Ind}_H^G \chi_f$ . The theorem is that this representation is always irreducible; its isomorphism class depends only on the  $G$ -orbit of  $f$ ; and, finally, every  $\rho \in \widehat{G}$  arises in this way from a unique  $G$ -orbit  $\Omega \subset \mathfrak{g}^*$ . This description of  $\rho_f$  is then used to prove Kirillov's character formula (or a suitable analogue thereof).

In reality, one needs to be more careful with uniformly powerful pro- $p$ -groups when  $p = 2$ . The problem that arises here is that an element  $f \in \mathfrak{g}^*$  which is trivial on  $[\mathfrak{g}, \mathfrak{g}]$  may not induce a 1-dimensional character of  $G$ . (We thank A. Jaikin-Zapirain for explaining this to us.) Thus in this case the approach to the orbit method has to be somewhat modified (cf. [JZ06]); however, the basic idea remains the same.

An important feature of all four situations mentioned above is that both  $\widehat{G}$  and  $\mathfrak{g}^*/G$  are equipped with a natural topology. The topology on the former is the so-called **Fell topology** (see §3.2). The topology on the latter is the quotient of the standard (compact-open) topology on  $\mathfrak{g}^*$ . Moreover, in all four cases the orbit method bijection turns out to be a homeomorphism. This is a non-trivial result which has useful applications. For an interesting application in the  $p$ -adic setting we refer the reader to [GK92]. In the setting of real Lie groups this statement was originally conjectured by Kirillov, who also proved that the bijection  $\mathfrak{g}^*/G \rightarrow \widehat{G}$  is continuous. The proof that this bijection is also open is substantially more difficult, and was given by Ian Brown about 10 years later in [Br73]. While it may be possible to adapt Brown's argument to a  $p$ -adic nilpotent Lie group  $G$  (to the best of our knowledge, this has never been done), we present in Section 3 a completely different proof (following a suggestion of V. Drinfeld) which is based on the fact that  $G$  is an increasing union of a sequence of open uniform pro- $p$ -subgroups (see Lemma 3.2); this is the main new result of our paper. Our proof seems to be much shorter and more transparent than Brown's proof, and we hope that it is easier to understand. On the other hand, it is not clear to us whether this approach has an analogue for real Lie groups.

Another new result in our paper is a theorem we call the “abstract orbit method”. It arose from an approach to the orbit method for finite nilpotent groups (of sufficiently small nilpotence class) that we also learned from V. Drinfeld. It was natural to try to see if this approach can be extended to uniform pro- $p$ -groups, and, more generally, to find the minimal set of assumptions under which this method can be used. The answer is given in Section 1, and in Section 2 we show that our “abstract orbit method” can indeed be used to classify (continuous) complex irreducible representations of  $p$ -groups of nilpotence class  $< p$  and of uniform pro- $p$ -groups (with a certain modification for  $p = 2$ ). The main difference with the classical approach is that we never mention polarizations. In particular, in the abstract setting one does not even need a Lie bracket on  $\mathfrak{g}$ . Instead, we prove directly that a suitable analogue of Kirillov’s character formula produces a collection of functions on the group, parameterized by the coadjoint orbits, which turn out to be precisely the irreducible characters of the group.

This approach has its advantages and disadvantages. The main disadvantage is that, unlike the classical one, our method of constructing irreducible characters cannot be “upgraded” to yield a construction of irreducible representations. On the other hand, it appears to be more straightforward, since one always works directly with irreducible characters, whereas the motivation behind the notion of a polarization comes from areas of mathematics outside of representation theory. However, a much more significant advantage is that the method explained in our paper has an analogue in the geometric representation theory for unipotent groups, whereas the classical method does not have such an analogue (at least not in any obvious sense), for, in the geometric setting polarizations cease to exist in general. A proper discussion of this remark is beyond the scope of our paper, and instead we refer the reader to [DB06].

ACKNOWLEDGEMENTS. This paper owes its existence to lectures of Vladimir Drinfeld and our private discussions with him. In particular, he explained to us the approach to the orbit method for finite nilpotent groups which gave rise to our “abstract orbit method” theorem. He also motivated our main result by asking if an analogue of Brown’s theorem for  $p$ -adic nilpotent Lie groups can be proved using the orbit method for uniformly powerful pro- $p$ -groups.

In addition, we would like to thank Michael Geline and George Glauberman who told us about Lazard's construction in the setting of  $p$ -groups and uniformly powerful pro- $p$ -groups, and suggested references [Khu98] and [DDMS], respectively. We are also grateful to Ben Wieland for referring us to [Ho77], to Adam Logan for drawing our attention to [JZ06], and especially to Andrei Jaikin-Zapirain for helpful e-mail correspondence and for pointing out a mistake in an earlier version of our paper.

**1. Abstract orbit method**

1.1. THE STATEMENT. For every profinite group  $\Pi$  we denote by  $\mu_\Pi$  the unique Haar measure on  $\Pi$  such that  $\mu_\Pi(\Pi) = 1$ . We define the **convolution** of two complex-valued  $L^2$ -functions  $f_1$  and  $f_2$  on  $\Pi$  by the formula

$$(f_1 * f_2)(\gamma) = \int_G f_1(h)f_2(h^{-1}\gamma) d\mu_\Pi(h), \quad \gamma \in \Pi.$$

We write  $\text{Fun}(\Pi)$  for the space of complex-valued functions on  $\Pi$  that are bi-invariant with respect to a sufficiently small open subgroup of  $\Pi$ . It is clear that  $\text{Fun}(\Pi) \subseteq L^2(\Pi)$  is closed under convolution, which makes  $\text{Fun}(\Pi)$  an associative  $\mathbb{C}$ -algebra (it is unital if and only if  $\Pi$  is finite, and commutative if and only if  $\Pi$  is commutative). The subspace  $\text{Fun}(\Pi)^\Pi \subseteq \text{Fun}(\Pi)$  of  $\Pi$ -invariant functions, where  $\Pi$  acts on itself by conjugation, is also closed under convolution (see Lemma 1.3), and in fact coincides with the **center** of  $\text{Fun}(\Pi)$ ; in particular,  $\text{Fun}(\Pi)^\Pi$  is always commutative.

**THEOREM 1.1** (Abstract orbit method): *Let  $G$  be a profinite group, and suppose that there exist an abelian profinite group  $\mathfrak{g}$  and a homeomorphism  $\exp : \mathfrak{g} \rightarrow G$  such that the following two conditions hold:*

- (i) *for each  $g \in G$ , the map  $\text{Ad } g : \mathfrak{g} \rightarrow \mathfrak{g}$  given by  $x \mapsto \log(g \exp(x)g^{-1})$  is a group automorphism, where we write  $\log$  for  $\exp^{-1}$ ; and*
- (ii) *the pullback map  $\exp^* : \text{Fun}(G)^G \xrightarrow{\cong} \text{Fun}(\mathfrak{g})^G$  commutes with convolution.*

*Then each  $G$ -orbit  $\Omega \subset \mathfrak{g}^*$  is finite, and there is a bijection between  $\mathfrak{g}^*/G$  and  $\widehat{G}$  such that the irreducible character  $\chi$  of  $G$  corresponding to an orbit  $\Omega \subset \mathfrak{g}^*$  is given by*

$$(1.1) \quad \chi(e^x) = |\Omega|^{-1/2} \sum_{f \in \Omega} f(x).$$

Here, as in the introduction,  $\mathfrak{g}^*$  denotes the Pontryagin dual of  $\mathfrak{g}$ , which, since  $\mathfrak{g}$  is compact, coincides with the group of continuous homomorphisms of  $\mathfrak{g}$  into  $\mathbb{C}^\times$ , and has the discrete topology. The action of  $G$  on  $\mathfrak{g}^*$  is induced by its action on  $\mathfrak{g}$  via  $\text{Ad}$ . Note that every finite group can be viewed as a profinite one (with the discrete topology), so our definitions and the theorem are valid for finite  $G$  and  $\mathfrak{g}$  as well.

*Remarks 1.2:* (1) In practice, if one wants to apply Theorem 1.1 to a specific group  $G$ , the main difficulty lies in verifying assumption (ii), as we will see in Section 2.

(2) As we have already noted in the introduction, one should observe that  $\mathfrak{g}$  is not required to have a Lie bracket in the statement of the theorem. Unfortunately, we do not know of any example where the assumption of the theorem is satisfied for some profinite group  $G$ , but  $\mathfrak{g}$  does not arise from some sort of a Lie algebra construction. It would be very interesting to find such an example.

(3) Formula (1.1) implies that  $|\Omega|^{1/2} = \chi(1)$  is an integer for every  $\Omega \in \mathfrak{g}^*/G$ , i.e., the order of every coadjoint orbit is a full square. In the generality of the theorem, this is the only proof of this fact known to us.

1.2. AUXILIARY RESULTS. In this section we fix  $G$ ,  $\mathfrak{g}$  and  $\exp$  satisfying the assumptions of the theorem. To simplify notation, we write  $X_G = \text{Fun}(G)^G$ . Thus  $X_G$  is the set of all complex-valued **class functions**  $f$  on  $G$  such that there exists a normal open subgroup  $K$  of  $G$  (depending on  $f$ ) satisfying

$$f(gk) = f(kg) = f(g), \quad \text{for all } g \in G, \quad \text{for all } k \in K.$$

Similarly, we will write  $X_{\mathfrak{g}} = \text{Fun}(\mathfrak{g})^G$ , and we write  $gxg^{-1}$  in place of  $(\text{Ad } g)(x)$ .

LEMMA 1.3: *We have  $X_G \subseteq L^2(G)$ , and  $X_G$  is an algebra with respect to convolution. Also, if  $\chi$  is the character of a continuous complex irreducible representation  $\rho$  of  $G$ , then  $(\dim \rho) \cdot \chi$  is an indecomposable idempotent of  $X_G$ , and every indecomposable idempotent of  $X_G$  has this form.*

Let us recall that an **indecomposable idempotent** of a commutative ring  $A$  is a non-zero idempotent  $e \in A$  (i.e.,  $e \neq 0$  and  $e^2 = e$ ) which cannot be written as  $e = e_1 + e_2$  for non-zero idempotents  $e_1, e_2 \in A$  satisfying  $e_1 \cdot e_2 = 0$ .

*Proof.* For any  $f_1, f_2 \in X_G$  let  $K$  be a normal open subgroup of  $G$  such that both  $f_1$  and  $f_2$  are constant on the cosets of  $K$  in  $G$ . (Clearly, such a  $K$  exists.) Then  $f_1$  and  $f_2$  can be considered as class functions on  $G/K$  (denoted respectively as  $\bar{f}_1$  and  $\bar{f}_2$ ), and for each  $g \in G$ , we have

$$(f_1 * f_2)(g) = (\bar{f}_1 * \bar{f}_2)(\bar{g}),$$

where  $\bar{g}$  denotes the image of  $g$  in  $G/K$ . On the other hand, it is well-known that every complex irreducible representation of  $G$  is finite dimensional (because  $G$  is compact), and hence has finite kernel (because  $G$  is totally disconnected). This implies that it is enough to prove the lemma for a finite group  $G$ .

Let  $G$  be finite. Then clearly the set  $\text{Fun}(G)$  of all functions on  $G$  is an algebra with convolution as multiplication and  $X_G \subseteq \text{Fun}(G)$  is the set of all class functions on  $G$ . Define the map  $\psi : \mathbb{C}[G] \rightarrow \text{Fun}(G)$  via

$$\psi(g)(h) = \begin{cases} 0, & g \neq h \\ |G|, & g = h \end{cases} \quad g, h \in G,$$

and for any  $x = \sum_{g \in G} n_g g$  ( $n_g \in \mathbb{C}$ ),

$$\psi(x) = \sum_{g \in G} \bar{n}_g \psi(g),$$

where  $\bar{n}_g$  denotes the complex conjugate of  $n_g$ . It is easy to see that  $\psi$  is a ring isomorphism and that the inverse image of  $X_G$  in  $\mathbb{C}[G]$  is center  $ZG$  of  $\mathbb{C}[G]$ . This implies that  $X_G \subseteq \text{Fun}(G)$  is a subalgebra and that  $e \in ZG$  is an indecomposable idempotent if and only if  $\psi(e)$  is one.

Recall that there is a one-to-one correspondence between indecomposable idempotents of  $ZG$  and irreducible representations of  $G$ , such that for every indecomposable idempotent  $e \in ZG$ , the corresponding irreducible representation  $\rho_e$  of  $G$  has the property that the left regular representation of  $G$  on  $e \cdot \mathbb{C}[G]$  is isomorphic to a multiple of  $\rho_e$ . Thus, it is enough to show that if  $e \in ZG$  is an indecomposable idempotent corresponding to an irreducible representation  $\rho_e$  of  $G$ , then  $(\dim \rho_e)^{-1} \cdot \psi(e)$  is the character of  $\rho_e$ . Moreover, since  $\psi(e)$  is a class function, it is enough to show that for any two indecomposable idempotents  $e, \tilde{e} \in ZG$  we have

$$(1.2) \quad \langle \psi(e), \chi_{\tilde{e}} \rangle = \begin{cases} 0, & e \neq \tilde{e} \\ \dim \rho_e, & e = \tilde{e} \end{cases},$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $L^2(G) = \text{Fun}(G)$  and  $\chi_{\tilde{e}}$  is the character of  $\rho_{\tilde{e}}$ .

Let  $e = \sum_{g \in G} n_g g, n_g \in \mathbb{C}$ . Then

$$(1.3) \quad \langle \psi(e), \chi_{\tilde{e}} \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \overline{\psi(e)(g)} \cdot \chi_{\tilde{e}}(g) = \sum_{g \in G} n_g \chi_{\tilde{e}}(g) = \chi_{\tilde{e}}(e),$$

where by extending  $\chi_{\tilde{e}}$  by linearity we consider  $\chi_{\tilde{e}}$  as a function on  $\mathbb{C}[G]$ . Now let  $V_{\tilde{e}}$  be a representation space for  $\rho_{\tilde{e}}$ . Then, as was mentioned above, for some  $n \in \mathbb{N}$  there exists a  $\mathbb{C}[G]$ -module isomorphism

$$\tilde{e} \cdot \mathbb{C}[G] \cong V_{\tilde{e}}^n,$$

hence

$$\chi_{\tilde{e}}(e) = \begin{cases} 0, & e \neq \tilde{e} \\ \dim \rho_e, & e = \tilde{e} \end{cases}$$

which together with (1.3) gives (1.2). ■

LEMMA 1.4: *Every  $G$ -orbit in  $\mathfrak{g}^*$  is finite.*

*Proof.* Fix  $f \in \mathfrak{g}^*$ . Since  $\mathfrak{g}$  is profinite and  $f : \mathfrak{g} \rightarrow \mathbb{C}^\times$  is continuous, it follows that  $f$  has finite image, hence  $\text{Ker } f$  is open. Thus there exists an open normal subgroup  $K \subseteq G$  such that  $\log(K) \subseteq \text{Ker } f$ . Hence every element in the  $G$ -orbit of  $f$  vanishes on  $\log(K)$ . But the set of elements of  $\mathfrak{g}^*$  that vanish on  $\log(K)$  is finite, since there exists an open additive subgroup  $\mathfrak{a} \subseteq \mathfrak{g}$  such that  $\mathfrak{a} \subseteq \log(K)$ , and  $(\mathfrak{g}/\mathfrak{a})^*$  is finite. ■

LEMMA 1.5: *Write  $\mu = \mu_{\mathfrak{g}}$ . For each  $\Omega \in \mathfrak{g}^*/G$  define the following function on  $\mathfrak{g}$*

$$\chi_\Omega(x) = \frac{1}{|\Omega|^{1/2}} \sum_{f \in \Omega} f(x), \quad x \in \mathfrak{g}.$$

*Then  $\chi_\Omega \in X_{\mathfrak{g}}$ , and for any two  $\Omega, \Omega' \in \mathfrak{g}^*/G$  we have*

$$\langle \chi_\Omega, \chi_{\Omega'} \rangle = \begin{cases} 1, & \Omega = \Omega' \\ 0, & \Omega \neq \Omega' \end{cases}$$

*where  $\langle \chi_\Omega, \chi_{\Omega'} \rangle = \int_{\mathfrak{g}} \chi_\Omega(x) \overline{\chi_{\Omega'}(x)} d\mu(x)$ .*

*Proof.* It is clear that  $\chi_\Omega \in X_{\mathfrak{g}}$ . Moreover, the functions  $f \in \mathfrak{g}^*$  are known to be orthonormal with respect to the inner product  $\langle \cdot, \cdot \rangle$ . Indeed, this is simply the orthogonality of irreducible characters for the compact abelian group  $\mathfrak{g}$ .



This implies in particular that the  $L^2$  norm of the function  $\sum_{f \in \Omega} f$  is equal to  $|\Omega|^{1/2}$ . Since distinct orbits are disjoint, the statement of the lemma follows immediately. ■

1.3. PROOF OF THEOREM 1.1. Let us write  $L^2(\mathfrak{g}^*)$  for the space of square-summable functions on  $\mathfrak{g}^*$  (where  $\mathfrak{g}^*$  is equipped with the counting measure). Recall that the Fourier transform provides an isomorphism

$$\mathcal{F} : L^2(\mathfrak{g}) \xrightarrow{\simeq} L^2(\mathfrak{g}^*),$$

$$(\mathcal{F}f)(\phi) = \int_{\mathfrak{g}} f(h)\overline{\phi(h)}d\mu(h), \quad f \in L^2(\mathfrak{g}), \phi \in \mathfrak{g}^*,$$

which intertwines convolution with pointwise multiplication (whenever the two operations are defined). Let  $\chi_\Omega$  be the functions on  $\mathfrak{g}$  defined in Lemma 1.5. Given  $\Omega \in \mathfrak{g}^*/G$ , we now show that  $\mathcal{F}(|\Omega|^{1/2} \cdot \chi_\Omega)$  is the characteristic function of  $\Omega$ . Let  $\Omega' \in \mathfrak{g}^*/G$  be a second orbit (possibly the same as  $\Omega$ ), and let  $\phi \in \Omega'$ . Then we have

$$\mathcal{F}(\chi_\Omega)(\phi) = \int_{\mathfrak{g}} \chi_\Omega(x)\overline{\phi(x)}d\mu(x).$$

This implies that

$$\begin{aligned} \langle \chi_\Omega, \chi_{\Omega'} \rangle &= \int_{\mathfrak{g}} \chi_\Omega(x)\overline{\chi_{\Omega'}(x)}d\mu(x) \\ &= \frac{1}{|\Omega'|^{1/2}|G\phi|} \sum_{g \in G} \int_{\mathfrak{g}} \chi_\Omega(x)\overline{\phi(g^{-1}xg)}d\mu(x) \\ &= \frac{|G|}{|\Omega'|^{1/2}|G\phi|} \int_{\mathfrak{g}} \chi_\Omega(x)\overline{\phi(x)}d\mu(x) = |\Omega'|^{1/2} \cdot \mathcal{F}(\chi_\Omega)(\phi). \end{aligned}$$

Thus, by Lemma 1.5,  $\mathcal{F}(|\Omega|^{1/2} \cdot \chi_\Omega)$  is the characteristic function of  $\Omega$ , and is therefore an indecomposable idempotent in the algebra of  $G$ -invariant functions on  $\mathfrak{g}^*$  with respect to pointwise multiplication. Hence  $|\Omega|^{1/2} \cdot \chi_\Omega$  is an indecomposable idempotent of the algebra  $X_G$  (with respect to convolution), which together with Lemma 1.3 proves the correspondence between irreducible representations of  $G$  and coadjoint orbits of  $\mathfrak{g}^*$ . Furthermore, if  $\chi$  is the irreducible character of  $G$  corresponding to  $\Omega$ , then

$$|\Omega|^{1/2} \cdot \chi_\Omega = \dim \rho \cdot \exp^*(\chi)$$

by Lemma 1.3, hence by evaluating these functions at 0 we see that  $|\Omega|^{1/2} = \dim \rho$ , and consequently  $\chi_\Omega = \exp^*(\chi)$ . This completes the proof of Theorem 1.1.

## 2. Application to (pro-) $p$ -groups

2.1. NOTATION AND TERMINOLOGY. From now on,  $p$  will denote a fixed prime number. Even though there exists a group-theoretic definition of a uniformly powerful (or, for brevity, “uniform”) pro- $p$ -group [DDMS], for our purposes it is more convenient to use the Lie-theoretic definition, which is also more transparent. We define a **uniform Lie algebra** to be a Lie algebra over the ring  $\mathbb{Z}_p$  of  $p$ -adic integers which is free of finite rank as a  $\mathbb{Z}_p$ -module and satisfies  $[\mathfrak{g}, \mathfrak{g}] \subseteq p \cdot \mathfrak{g}$  (respectively,  $[\mathfrak{g}, \mathfrak{g}] \subseteq 4 \cdot \mathfrak{g}$  when  $p = 2$ ). Given a uniform Lie algebra  $\mathfrak{g}$ , we equip it with the topology induced by the standard topology on  $\mathbb{Z}_p$ , and we define a topological group  $G := \exp \mathfrak{g}$  to be the underlying topological space of  $\mathfrak{g}$  equipped with a group operation given by the **Campbell-Hausdorff series**

$$(2.1) \quad CH(x, y) = \log(\exp(x) \exp(y)) = \sum_{i=1}^{\infty} CH_i(x, y).$$

*Remarks 2.1:* (1) A priori,  $CH(x, y)$  is viewed as an element of  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ , the algebra of formal non-commutative power series in the variables  $x$  and  $y$  with coefficients in  $\mathbb{Q}$ , and  $CH_i(x, y)$  denotes its homogeneous component of (total) degree  $i$ . However, it is well-known that  $CH(x, y)$  is, in fact, a **formal Lie series**, which means that each term  $CH_i(x, y)$  lies in the Lie subalgebra of  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  generated by  $x$  and  $y$ .

(2) Since the coefficients of  $CH(x, y)$  involve positive powers of  $p$  in the denominator, it is not immediately obvious that  $CH(x, y)$  can even be evaluated in a uniform Lie algebra  $\mathfrak{g}$  term-by-term. However, Michel Lazard proved, cf. [DDMS], that the condition imposed on  $\mathfrak{g}$  guarantees that for each  $x, y \in \mathfrak{g}$ , we have  $CH_i(x, y) \in \mathfrak{g} \subseteq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{g}$ , and, in addition, the series  $CH(x, y)$  converges uniformly on  $\mathfrak{g}$  and makes it a topological group. This result justifies our construction of  $G = \exp \mathfrak{g}$ .

*Definition 2.2:* A **uniform pro- $p$ -group** is a profinite group  $G$  which is isomorphic to  $\exp \mathfrak{g}$  for some uniform Lie algebra  $\mathfrak{g}$ . If  $G \cong \exp \mathfrak{g}$  is such a group, we will fix an isomorphism  $\exp \mathfrak{g} \xrightarrow{\cong} G$  and denote the underlying map of sets by  $\exp : \mathfrak{g} \rightarrow G$ . By abuse of notation, we will also write  $G = \exp \mathfrak{g}$  and  $\mathfrak{g} = \text{Lie}(G)$ .

Suppose now that  $\mathfrak{g}$  is a finite Lie ring (i.e., a Lie algebra over  $\mathbb{Z}$ ) whose order is a power of  $p$ , and such that  $\mathfrak{g}$  is nilpotent of nilpotence class  $< p$ . (This

means that any iterated commutator of length  $\geq p$  vanishes in  $\mathfrak{g}$ .) In this case it is rather easy to check that the Lie series  $CH(x, y)$  can be evaluated in  $\mathfrak{g}$  term-by-term, and makes  $\mathfrak{g}$  a  $p$ -group (there is no issue of convergence of  $CH$  in this setting). Again, we denote this  $p$ -group by  $\exp \mathfrak{g}$ . Michel Lazard proved, cf. [Khu98], that every  $p$ -group  $G$  of nilpotence class  $< p$  arises in this way from a unique  $\mathfrak{g}$ . In this situation we will also write  $\mathfrak{g} = \text{Lie}(G)$ , and the underlying map of a fixed isomorphism  $\exp \mathfrak{g} \xrightarrow{\cong} G$  will be denoted by  $\exp : \mathfrak{g} \rightarrow G$ , just as for uniform pro- $p$ -groups. We will write  $\log = \exp^{-1}$ .

**2.2. AUXILIARY RESULTS ON FORMAL LIE SERIES.** Throughout the rest of the section  $G$  will denote a  $p$ -group of nilpotence class  $< p$  or a uniform pro- $p$ -group, and  $\mathfrak{g} = \text{Lie}(G)$  its Lie algebra. In both cases we have the exponential map  $\exp : \mathfrak{g} \rightarrow G$ , which is a homeomorphism, and assumption (i) of Theorem 1.1 is satisfied in this situation. In fact, in both cases it is known that Lazard's construction is functorial; in particular, a continuous set-theoretic bijection  $\mathfrak{g} \rightarrow \mathfrak{g}$  is a Lie algebra automorphism if and only if it is an automorphism of the group  $\exp \mathfrak{g}$ . Another fact that will often be used implicitly in what follows is that if  $x \in \mathfrak{g}$  and  $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}$  denotes the additive map  $y \mapsto [x, y]$ , then  $e^{\text{ad } x} = \text{Ad}(e^x)$  as automorphisms of  $\mathfrak{g}$ . In order to classify the continuous complex irreducible representations of  $G$  we would like to show that assumption (ii) of Theorem 1.1 holds in this setting as well. Unfortunately, as A. Jaikin-Zapirain pointed out to us, this is sometimes false when  $p = 2$  and  $G$  is a uniform pro-2-group; thus this case needs to be dealt with separately (see §2.4).

Ignoring this issue for the moment, let us note that the main problem with verifying assumption (ii) arises from the fact that the convolution of functions on  $\mathfrak{g}$  is defined using the addition in  $\mathfrak{g}$ , while the convolution of functions on  $\exp \mathfrak{g}$  is defined using the multiplication in  $\exp \mathfrak{g}$ , or, equivalently, the Campbell–Hausdorff operation  $CH : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ . This problem is dealt with in a very natural way, shown to us by V. Drinfeld: we prove that one can write  $CH(x, y) = \tilde{x} + \tilde{y}$ , where  $\tilde{x}$  (respectively,  $\tilde{y}$ ) is a certain Lie series in the variables  $x, y$  which is conjugate to  $x$  (respectively, to  $y$ ). This formula implies that the two convolutions of conjugation-invariant functions on  $\mathfrak{g}$ , defined using addition and the operation  $CH$ , are in fact identical, which is the content of condition (ii) of Theorem 1.1. We should mention that in practice, however,

the realization of this idea for uniform pro- $p$ -groups involves certain technical difficulties.

We now turn to precise statements. Let  $E$  be a finite extension of  $\mathbb{Q}_p$ , let  $v_p$  denote the valuation on  $E$  normalized by  $v_p(p) = 1$ , and let  $\mathcal{K} \subseteq E$  be a subfield.

LEMMA 2.3: *Let  $H(x, y) = \sum_{n=1}^{\infty} H_n(x, y) \in \mathcal{K}\langle\langle x, y \rangle\rangle$  be a formal Lie series, where  $H_n(x, y)$  is homogeneous of degree  $n$ , such that  $H_1(x, y) = x + y$  and  $v_p(H_n) \geq -\frac{n-2}{p-1}$  for all  $n \geq 2$ . Then there exist formal Lie series  $\phi = \phi(x, y)$ ,  $\psi = \psi(x, y) \in \mathcal{K}\langle\langle x, y \rangle\rangle$  such that*

$$(2.2) \quad H(x, y) = e^{\text{ad } \phi(x, y)}(x) + e^{\text{ad } \psi(x, y)}(y),$$

and if  $\phi_n, \psi_n$  denote the degree  $n$  homogeneous components of  $\phi, \psi$ , respectively, then  $v_p(\phi_n) \geq -\frac{n-1}{p-1}$  and  $v_p(\psi_n) \geq -\frac{n-1}{p-1}$  for all  $n \geq 1$ .

Here, by abuse of notation, we write  $v_p(H_n)$  (respectively,  $v_p(\phi_n)$  and  $v_p(\psi_n)$ ) for the minimum among the valuations of all coefficients of  $H_n$  (respectively,  $\phi_n$  and  $\psi_n$ ).

*Proof.* It is easy to see that one can construct the series  $\phi$  and  $\psi$  inductively. Namely, for each  $n \geq 0$  let us compare the homogeneous components of degree  $n + 1$  on both sides of (2.2). For  $n = 0$  there is nothing to check, thanks to the assumption that  $H_1(x, y) = x + y$ . For each  $n \geq 1$ , we may assume that all  $\phi_j, \psi_j$  with  $j < n$  have already been found and satisfy  $v_p(\phi_j), v_p(\psi_j) \geq -\frac{j-1}{p-1}$ . In order to find  $\phi_n$  and  $\psi_n$  we have to solve an equation of the form

$$(2.3) \quad [\phi_n, x] + [\psi_n, y] + (\text{something known}) = H_{n+1}(x, y),$$

where “something known” is a sum of expressions of the form

$$(2.4) \quad \frac{1}{k!} \cdot [\phi_{j_1}, [\phi_{j_2}, [\dots [\phi_{j_k}, x] \dots ]]] \quad \text{or} \quad \frac{1}{k!} \cdot [\psi_{j_1}, [\psi_{j_2}, [\dots [\psi_{j_k}, y] \dots ]]]$$

with  $2 \leq k \leq n$  and  $j_1 + j_2 + \dots + j_k = n$ . It is well-known that  $v_p(k!) \leq \frac{k-1}{p-1}$  for all  $k \geq 1$  (with equality if  $k$  is a power of  $p$ ), which implies that the valuation of each of the expressions in (2.4) is at least

$$-\frac{k-1}{p-1} - \frac{j_1-1}{p-1} - \dots - \frac{j_k-1}{p-1} = -\frac{k-1 + j_1 + \dots + j_k - k}{p-1} = -\frac{n-1}{p-1}.$$

In addition, we have  $v_p(H_{n+1}) \geq -\frac{n-1}{p-1}$  by assumption. This immediately implies that there exist homogeneous Lie polynomials  $\phi_n = \phi_n(x, y)$ ,  $\psi_n =$

$\psi_n(x, y)$  of degree  $n$  which solve (2.3) and satisfy  $v_p(\phi_n), v_p(\psi_n) \geq -\frac{n-1}{p-1}$ , completing the induction. ■

This result suffices to prove the orbit method correspondence when  $p \geq 5$ . To treat the case  $p = 3$  we need the following variation

LEMMA 2.4: *In the situation of Lemma 2.3, assume that  $p = 3$  and that*

$$v_3(H_n) \geq -\frac{6n - 10}{7} \quad \text{for all } n \geq 2.$$

*Then the conclusion of Lemma 2.3 holds with  $v_3(\phi_n), v_3(\psi_n) \geq -\frac{6n-4}{7}$  for  $n \geq 1$ .*

*Proof.* We follow the proof of Lemma 2.3 almost word-for-word; the only step that needs to be changed is the estimation of the valuations of the coefficients of the expressions (2.4). We have  $v_3(k!) \leq (k - 1)/2$ , and therefore, by the induction assumption, each of the valuations in question is at least

$$-\frac{k - 1}{2} - \frac{6j_1 - 4}{7} - \dots - \frac{6j_k - 4}{7} = -\frac{k - 1}{2} - \frac{6n - 4k}{7} = -\frac{6n - 4}{7} + \frac{k - 1}{14}.$$

Since  $k - 1 > 0$ , this finishes the induction in the same way as before. ■

2.3. THE ORBIT METHOD WHEN  $p \geq 3$ . In this subsection we treat the orbit method for a group  $G$  which is either a  $p$ -group of nilpotence class  $< p$  or a uniform pro- $p$ -group with  $p \geq 3$ . Since the orbit method obviously works for commutative 2-groups, there is no harm in assuming that  $p \geq 3$  for finite  $G$  as well.

PROPOSITION 2.5: *Assume that  $p \geq 3$ , let  $G$  be as above, and let  $\mathfrak{g} = \text{Lie}(G)$ . Then there exist formal Lie series  $\phi(x, y), \psi(x, y) \in \mathbb{Q}\langle\langle x, y \rangle\rangle$  which can be evaluated term-by-term in  $\mathfrak{g}$ , converge uniformly for  $x, y \in \mathfrak{g}$  when  $\mathfrak{g}$  is uniform, and satisfy*

$$(2.5) \quad \log(\exp(x) \exp(y)) = e^{\text{ad } \phi(x,y)}(x) + e^{\text{ad } \psi(x,y)}(y) \quad \text{for all } x, y \in \mathfrak{g}.$$

*Proof.* Let us recall the Campbell–Hausdorff series  $CH(x, y) \in \mathbb{Q}\langle\langle x, y \rangle\rangle$  defined by (2.1). The key fact about the coefficients of  $CH(x, y)$  that we will need is the following result (see [DDMS], p. 123): for every prime  $p$ ,

$$(2.6) \quad v_p(CH_n) \geq -\frac{n - 1}{p - 1} \quad \text{for all } n \geq 1.$$

Suppose first that  $G$  is finite, and let  $H(x, y) \in \mathbb{Q}\langle\langle x, y \rangle\rangle$  denote the Lie polynomial obtained by discarding all homogeneous components of  $CH(x, y)$  of degree  $\geq p$ . Then  $G$  is isomorphic to  $\mathfrak{g}$  equipped with the operation given by  $H$ . Since  $v_p(H_n) \in \mathbb{Z}$ , it follows from (2.6) that all coefficients of  $H$  lie in  $\mathbb{Z}_p \cap \mathbb{Q}$ . Thus the assumption of Lemma 2.3 is satisfied with  $\mathcal{K} = \mathbb{Q}$ . Let  $\phi', \psi'$  denote formal Lie series satisfying the conclusion of the lemma, and let  $\phi, \psi$  be the Lie polynomials obtained from  $\phi'$  and  $\psi'$ , respectively, by discarding all homogeneous components of degrees  $\geq p - 1$ . (Note a slight change of our notation!) Since the valuation of each coefficient of  $\phi$  and  $\psi$  must be an integer, and since  $-\frac{n-1}{p-1} > -1$  for  $1 \leq n \leq p - 2$ , we see that the coefficients of  $\phi$  and  $\psi$  lie in  $\mathbb{Z}_p \cap \mathbb{Q}$ . Thus  $\phi$  and  $\psi$  can be evaluated in  $\mathfrak{g}$ . In addition, since  $\mathfrak{g}$  has nilpotence class  $\leq p - 1$ , the conclusion of Lemma 2.3 implies that (2.5) holds.

Next we assume that  $G$  is a uniform pro- $p$ -group. By abuse of notation, we define a “valuation”  $v_p : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{g} \rightarrow \mathbb{Z} \cup \{\infty\}$  by  $v_p(x) = \sup\{r \in \mathbb{Z} : p^{-r}x \in \mathfrak{g}\}$ . It is well-known that a series  $\sum_{n=1}^{\infty} x_n$  in  $\mathfrak{g}$  converges if and only if  $v_p(x_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Let us consider the case  $p = 3$ . Put  $H(x, y) = CH(x, y) \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ . It follows from (2.6) that  $v_3(H_2) \geq 0$  and  $v_3(H_n) \geq -\frac{n-1}{2}$  for all  $n \geq 3$ . Since  $-\frac{6n-10}{7} \leq -\frac{n-1}{2}$  for all  $n \geq 3$ , we see that  $H$  satisfies the assumption of Lemma 2.4 with  $\mathcal{K} = \mathbb{Q}$ . Let  $\phi, \psi$  denote the formal Lie series satisfying the conclusion of the lemma. Since  $\mathfrak{g}$  is uniform, we see that for all  $x, y \in \mathfrak{g}$  and all  $n \geq 1$ , we have

$$v_3(\phi_n(x, y)) \geq v_3(\phi_n) + n - 1 \geq n - 1 - \frac{6n - 4}{7} = \frac{n - 3}{7} > -1.$$

Therefore  $v_3(\phi_n(x, y)) \geq 0$ , which means that  $\phi_n(x, y)$  can be evaluated in  $\mathfrak{g}$  for all  $n \geq 1$ , and, in addition,  $v_3(\phi_n(x, y)) \rightarrow \infty$  as  $n \rightarrow \infty$  (independently of  $x, y$ ), which implies that the series  $\phi(x, y)$  converges uniformly in  $\mathfrak{g}$  for  $x, y \in \mathfrak{g}$ . Similarly, the series  $\psi(x, y)$  can be evaluated term-by-term and converges uniformly for all  $x, y \in \mathfrak{g}$ .

Finally, we consider the case  $p \geq 5$ . Here an additional small trick is needed. Put  $\mathcal{K} = E = \mathbb{Q}_p(\sqrt{p})$  and  $H(x, y) = 1/\sqrt{p} \cdot CH(\sqrt{p}x, \sqrt{p}y) \in \mathcal{K}\langle\langle x, y \rangle\rangle$ . Then (2.6) implies that  $v_p(H_n) = n/2 + v_p(CH_n) - 1/2 \geq 0$  for all  $n \geq 1$ , and we have  $H_1(x, y) = x + y$ , which shows that Lemma 2.3 applies to  $H(x, y)$ . Changing notation again, we let  $\phi'(x, y), \psi'(x, y)$  be the formal Lie series satisfying the

conclusion of the lemma. Thus

$$\frac{1}{\sqrt{p}} \log(\exp(\sqrt{p}x) \exp(\sqrt{p}y)) = e^{\text{ad } \phi'(x,y)}(x) + e^{\text{ad } \psi'(x,y)}(y),$$

which after a change of variables  $z = \sqrt{p}x, w = \sqrt{p}y$  can be rewritten as

$$\log(\exp(z) \exp(w)) = e^{\text{ad } \phi(z,w)}(z) + e^{\text{ad } \psi(z,w)}(w),$$

where we have put  $\phi(z, w) = \phi'(z/\sqrt{p}, w/\sqrt{p})$  and  $\psi(z, w) = \psi'(z/\sqrt{p}, w/\sqrt{p})$ . The problem is that, a priori, the coefficients of  $\phi$  and  $\psi$  lie in  $\mathcal{K} = \mathbb{Q}_p(\sqrt{p})$ . However, this is easy to fix as follows. Let us introduce a  $\mathbb{Z}/2\mathbb{Z}$ -grading on  $\mathcal{K}\langle\langle x, y \rangle\rangle$  by assigning degree 0 to every element of  $\mathbb{Q}_p$ , and assigning degree 1 to  $x, y$  and  $\sqrt{p}$ . With this convention, it is clear that  $H(x, y)$  is **purely odd** (i.e., each  $H_n(x, y)$  is odd). By looking at the proof of Lemma 2.3, it is easy to see that the formal Lie series  $\phi'$  and  $\psi'$  can be chosen to be purely even. This implies that  $\phi(z, w)$  and  $\psi(z, w)$  have coefficients in  $\mathbb{Q}_p$ .

The rest of the proof is the same as before. For all  $z, w \in \mathfrak{g}$  and all  $n \geq 1$ , we have

$$\begin{aligned} v_p(\phi_n(z, w)) &\geq v_p(\phi_n) + (n - 1) = v_p(\phi'_n) - \frac{n}{2} + (n - 1) \\ &\geq -\frac{n - 1}{p - 1} - \frac{n}{2} + n - 1 \geq \frac{n - 3}{4} > -1, \end{aligned}$$

where we have used<sup>2</sup> the assumption  $p \geq 5$ . Thus  $\phi(z, w)$  can be evaluated term-by-term in  $\mathfrak{g}$  and converges uniformly for all  $z, w \in \mathfrak{g}$ , and similarly for  $\psi(z, w)$ . ■

**THEOREM 2.6:** *Assume that  $p \geq 3$ , let  $G$  be either a  $p$ -group of nilpotence class  $< p$  or a uniform pro- $p$ -group, and let  $\mathfrak{g} = \text{Lie}(G)$ . Then there exists a bijection  $\Omega \longleftrightarrow \chi_\Omega$  between  $G$ -orbits  $\Omega \subset \mathfrak{g}^*$  and characters of representations  $\rho \in \widehat{G}$  such that Kirillov's character formula holds:*

$$(2.7) \quad \chi_\Omega(e^x) = |\Omega|^{-1/2} \cdot \sum_{f \in \Omega} f(x) \quad \text{for all } x \in \mathfrak{g}.$$

*Proof.* We will show that hypothesis (ii) of Theorem 1.1 holds for  $\exp : \mathfrak{g} \rightarrow G$ . Let  $\phi(x, y)$  and  $\psi(x, y)$  be the formal Lie series satisfying the conclusion of Proposition 2.5. Then we obtain a continuous map of  $\mathfrak{g} \times \mathfrak{g}$  to itself given by  $(x, y) \mapsto (\tilde{x}, \tilde{y}) = (e^{\text{ad } \phi(x,y)}(x), e^{\text{ad } \psi(x,y)}(y))$ . (This is a slight abuse of notation since  $\tilde{x}$  depends on both  $x$  and  $y$ , and so does  $\tilde{y}$ .) This map satisfies the

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<sup>2</sup> Note that this argument would fail if  $p = 3$ : this is why we need Lemma 2.4.

properties mentioned in §2.2: on the one hand,  $\tilde{x}$  and  $\tilde{y}$  are conjugate to  $x$  and  $y$ , respectively, and on the other hand, we have  $CH(x, y) = \tilde{x} + \tilde{y}$  for all  $x, y \in \mathfrak{g}$ . We will now use this information to show that  $\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2)$  for all  $f_1, f_2 \in \text{Fun}(G)^G$ .

Let us first assume that  $G$  is finite. Then, due to the  $G$ -invariance of  $f_1, f_2$ , we have

$$(2.8) \quad (\exp^*(f_1 * f_2))(z) \stackrel{\text{def}}{=} \frac{1}{|\mathfrak{g}|} \sum_{x, y \in \mathfrak{g}: \tilde{x} + \tilde{y} = z} f_1(e^x) f_2(e^y) = \frac{1}{|\mathfrak{g}|} \sum_{x, y \in \mathfrak{g}: \tilde{x} + \tilde{y} = z} f_1(e^{\tilde{x}}) f_2(e^{\tilde{y}})$$

for all  $z \in \mathfrak{g}$ . On the other hand,

$$(2.9) \quad (\exp^*(f_1) * \exp^*(f_2))(z) = \frac{1}{|\mathfrak{g}|} \sum_{x, y \in \mathfrak{g}: x + y = z} f_1(e^x) f_2(e^y).$$

Thus it only remains to show that the map  $(x, y) \mapsto (\tilde{x}, \tilde{y})$  is a bijection of  $\mathfrak{g} \times \mathfrak{g}$  onto itself. However, this map is of the form

$$(x, y) \mapsto (x + A(x, y), y + B(x, y)),$$

where  $A$  and  $B$  are Lie polynomials whose homogeneous components have degrees  $\geq 2$ . Using induction on the nilpotence class of  $\mathfrak{g}$ , it is easy to check that this map is injective (for the induction step, let  $\mathfrak{z}$  be the center of  $\mathfrak{g}$  and note that the map descends to a map of  $(\mathfrak{g}/\mathfrak{z}) \times (\mathfrak{g}/\mathfrak{z})$  to itself). Therefore, it is bijective because  $\mathfrak{g}$  is finite.

If  $G$  is uniform the argument is similar. We only need to recall that by the definition of  $\text{Fun}(G)$ , there exists  $r \in \mathbb{N}$  such that  $f_1$  and  $f_2$  are bi-invariant with respect to the open subgroup  $G^{p^r} = \exp(p^r \mathfrak{g})$ . Moreover,  $\exp$  descends to a bijection of  $\mathfrak{g}/p^r \mathfrak{g}$  onto  $G/G^{p^r}$ , and the map  $(x, y) \mapsto (\tilde{x}, \tilde{y})$  descends to a bijection of  $(\mathfrak{g}/p^r \mathfrak{g}) \times (\mathfrak{g}/p^r \mathfrak{g})$  onto itself. Thus equations (2.8) and (2.9) remain valid with  $\mathfrak{g}$  replaced by  $\mathfrak{g}/p^r \mathfrak{g}$ , and we see that  $\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2)$ , as desired. ■

2.4. UNIFORM PRO-2-GROUPS. In this subsection we assume that  $p = 2$ , fix a uniform pro-2-group  $G$ , and put  $\mathfrak{g} = \text{Lie}(G)$ . As we have already mentioned, the conclusion of Theorem 2.6 may fail in this case; indeed, even for an orbit  $\Omega \subset \mathfrak{g}^*$  of size 1 formula (2.7) may fail to define an irreducible character of  $G$  (cf. [JZ06]). In view of Theorem 1.1, this means that the pullback map  $\exp^* : \text{Fun}(G)^G \rightarrow \text{Fun}(\mathfrak{g}^*)^G$  may not commute with convolution. However, we do have the following weaker positive result.



- PROPOSITION 2.7: (a) Assume that  $[\mathfrak{g}, \mathfrak{g}] \subseteq 8 \cdot \mathfrak{g}$ . Given  $f_1, f_2 \in \text{Fun}(G)^G$ , we have  $\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2)$  provided either  $f_1$  or  $f_2$  is supported on  $G^2$ .
- (b) In general, we have  $\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2)$  for all  $f_1, f_2 \in \text{Fun}(G^2)^G$ .

*Proof.* (a) The argument is rather similar to the one used in the previous subsection. Consider the formal Lie series  $H(x, y) = 1/2 \cdot CH(2x, 2y)$ . We have  $H_1(x, y) = x + y$ , and it follows from (2.6) that  $v_2(H_n) \geq 0$  for all  $n \geq 1$ . Thus we can apply Lemma 2.3 with  $\mathcal{K} = \mathbb{Q}$ , and it yields formal Lie series  $\phi'(x, y), \psi'(x, y)$  satisfying

$$\frac{1}{2} \cdot \log(\exp(2x) \exp(2y)) = e^{\text{ad } \phi'(x,y)}(x) + e^{\text{ad } \psi'(x',y')}(y).$$

We make the change of variables  $z = 2x, w = 2y$  and rewrite the last equation as

$$\log(\exp(z) \exp(w)) = e^{\text{ad } \phi(z,w)}(z) + e^{\text{ad } \psi(z,w)}(w),$$

where  $\phi(z, w) = \phi'(\frac{z}{2}, \frac{w}{2})$  and  $\psi(z, w) = \psi'(\frac{z}{2}, \frac{w}{2})$ . Now if  $z, w \in \mathfrak{g}$ , then for all  $n \geq 1$ ,

$$(2.10) \quad \begin{aligned} v_2(\phi_n(z, w)) &\geq v_2(\phi_n) + 3(n - 1) = v_2(\phi'_n) - n + 3(n - 1) \\ &\geq -(n - 1) - n + 3(n - 1) = n - 2. \end{aligned}$$

Here we have used the assumption  $[\mathfrak{g}, \mathfrak{g}] \subseteq 8 \cdot \mathfrak{g}$ . Similarly,  $v_2(\psi_n(z, w)) \geq n - 2$  for all  $n \geq 1$ . This means that the series  $\sum_{n \geq 2} \phi_n(z, w)$  and  $\sum_{n \geq 2} \psi_n(z, w)$  can be evaluated term-by-term and converge uniformly for  $z, w \in \mathfrak{g}$ . Unfortunately, we cannot make sure that both  $\phi_1(z, w)$  and  $\psi_1(z, w)$  are defined in  $\mathfrak{g}$ , because by definition we must have  $[\phi_1(z, w), z] + [\psi_1(z, w), w] = \frac{1}{2}[z, w]$ . However, in view of the inductive construction of the series  $\phi'$  and  $\psi'$  used in the proof of Lemma 2.3, we may assume that, say,  $\phi_1(z, w) = 0$  and  $\psi_1(z, w) = z/2$ . This implies that  $\phi(z, w)$  is defined and converges uniformly in  $\mathfrak{g}$  for all  $z, w \in \mathfrak{g}$ , while  $\psi(z, w)$  is defined and converges uniformly in  $\mathfrak{g}$  for  $z \in 2\mathfrak{g}$  and  $w \in \mathfrak{g}$ .

The rest of the proof is as before. Put  $(\tilde{x}, \tilde{y}) = (e^{\text{ad } \phi(x,y)}(x), e^{\text{ad } \psi(x,y)}(y))$ . Then  $(x, y) \mapsto (\tilde{x}, \tilde{y})$  is a map from  $(2\mathfrak{g}) \times \mathfrak{g}$  to itself, and the argument used in the proof of Theorem 2.6 implies that  $\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2)$  if  $f_1, f_2 \in \text{Fun}(G)^G$  and  $f_1$  is supported on  $G^2$ . Since convolution of  $G$ -invariant functions is commutative, the same formula holds if instead  $f_2$  is supported on  $G^2$ , completing the proof of (a).

The proof of (b) is almost identical, except that (2.10) has to be replaced by the following estimate, which is valid whenever  $[\mathfrak{g}, \mathfrak{g}] \subseteq 4\mathfrak{g}$  and  $z, w \in 2\mathfrak{g}$ :

$$\begin{aligned} v_2(\phi_n(z, w)) &\geq v_2(\phi_n) + 2(n - 1) + n = v_2(\phi'_n) - n + 2(n - 1) + n \\ &\geq -(n - 1) + 2(n - 1) = n - 1 \geq 0. \end{aligned}$$

This means that  $\phi(z, w)$ , and similarly  $\psi(z, w)$ , can be evaluated in  $\mathfrak{g}$  term-by-term for all  $z, w \in 2\mathfrak{g}$ , and converges uniformly for these values of  $z, w$ . ■

We can now prove a version of the orbit method for uniform pro-2-groups which is weaker than Theorem 2.6, but suffices for some applications (see Section 3).

**THEOREM 2.8:** *Let  $G$  be a uniform pro-2-group and  $\mathfrak{g} = \text{Lie}(G)$ . For every  $G$ -orbit  $\Omega \subset (2\mathfrak{g})^*$ , let  $e'_\Omega \in \text{Fun}(2\mathfrak{g})^G$  denote the inverse Fourier transform of the characteristic function of  $\Omega$ , put  $e_\Omega = \log^*(e'_\Omega) \in \text{Fun}(G^2)^G$ , and define  $\widehat{G}_\Omega \subset \widehat{G}$  to be the collection of those  $\rho \in \widehat{G}$  on which  $e_\Omega$  acts nontrivially<sup>3</sup>. Then the following statements hold:*

- (a) *each  $\widehat{G}_\Omega$  is finite, and  $\widehat{G}$  is the disjoint union of the subsets  $\widehat{G}_\Omega$ ;*
- (b) *if  $\rho \in \widehat{G}_\Omega$  and  $\chi_\rho$  is its character, then  $\chi_\rho|_{G^2}$  is a multiple of  $e_\Omega$ .*

*Proof.* We use a modification of the argument that appeared in the proof of Theorem 1.1. By construction,  $e'_\Omega$  is an indecomposable idempotent in the algebra  $\text{Fun}(2\mathfrak{g})^G$  (with respect to the convolution defined using addition in  $\mathfrak{g}$ ), and Proposition 2.7(b) implies that  $e_\Omega$  is an indecomposable idempotent in  $\text{Fun}(G^2)^G$ . Now we can think of  $\text{Fun}(G^2)^G$  as a subalgebra of  $\text{Fun}(G^2)^{G^2}$  in the obvious way, as well as a subalgebra of  $\text{Fun}(G)^G$  using extension by zero. Therefore, we can write

$$e_\Omega = \sum_{i=1}^m e_i = \sum_{j=1}^n f_j,$$

where the  $e_i$ 's are indecomposable idempotents in  $\text{Fun}(G^2)^{G^2}$  and the  $f_j$ 's are indecomposable idempotents in  $\text{Fun}(G)^G$ . By the proof of Theorem 1.1, each  $e_i$  (respectively,  $f_j$ ) corresponds to some  $\pi_i \in \widehat{G^2}$  (respectively,  $\rho_j \in \widehat{G}$ ) whose character is a multiple of  $e_i$  (respectively,  $f_j$ ). It is clear that if  $\rho \in \widehat{G}$ , then  $\rho(e_\Omega) \neq 0$  if and only if  $\rho \cong \rho_j$  for some  $j$ , which implies that  $\widehat{G}_\Omega = \{\rho_1, \rho_2, \dots, \rho_n\}$  is finite, proving the first half of (a).

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<sup>3</sup> This means that the linear operator  $\rho(e_\Omega) := \int_G \overline{\rho(g)}\rho(g)d\mu_G(g)$  is non-zero.

Next, let  $\chi_i \in \text{Fun}(G)^G$  be the character of the induced representation  $\eta_i := \text{Ind}_{G^2}^G \pi_i$ . Since  $G^2$  is normal in  $G$ , it follows that  $\chi_i$  is supported on  $G^2$ , so that we can think of it as an element of  $\text{Fun}(G^2)^G$ , and, moreover,  $\chi_i$  is a positive integral multiple of the sum of elements in the orbit of  $e_i$  under the  $G$ -conjugation action. In particular,  $\chi_i * e_\Omega \neq 0$ , and Therefore,  $\chi * e_\Omega = \lambda_i \cdot e_\Omega$  for some  $\lambda_i \in \mathbb{C}^\times$ , because  $e_\Omega$  is a indecomposable idempotent in  $\text{Fun}(G^2)^G$ . Hence, we must have  $\chi_i * f_j = \lambda_i \cdot f_j$  for every  $j$ . Therefore, the  $\rho_j$ 's are precisely the irreducible constituents of  $\eta_i$ . Now the Frobenius reciprocity implies that for each  $1 \leq j \leq n$ , the  $\pi_i$ 's are precisely the irreducible constituents of  $\rho_j|_{G^2}$ , which proves part (b).

Finally, to finish the proof of (a), let  $\rho \in \widehat{G}$  be arbitrary, and let  $f \in \text{Fun}(G)^G$  denote the corresponding indecomposable idempotent. There exists a normal open subgroup  $K \subset G$  such that  $K \subseteq G^2$  and  $f$  is bi-invariant with respect to  $K$ . Therefore,  $f$  is the pullback of an indecomposable idempotent  $\bar{f}$  of  $\text{Fun}(G/K)^{G/K}$ . However, the natural inclusion  $\text{Fun}(G^2/K)^{G/K} \hookrightarrow \text{Fun}(G/K)^{G/K}$  is a homomorphism of unital algebras, which implies that  $\bar{f}$  is a summand of an indecomposable idempotent  $\bar{e}$  of  $\text{Fun}(G^2/K)^{G/K}$ . Let  $e \in \text{Fun}(G^2)^G$  be the pullback of  $\bar{e}$ ; it follows from Proposition 2.7(b) that  $e = e_\Omega$  for some  $G$ -orbit  $\Omega \subset (2\mathfrak{g})^*$ , and the proof is complete. ■

2.5. CONCLUDING REMARKS. The orbit method for uniform pro- $p$ -groups was first studied by Roger Howe [Ho77]; he used the classical approach based on the notion of a polarization. However, he did not treat the case  $p = 2$ , and his results for  $p \geq 3$  are weaker than our Theorem 2.6 in that he has to impose an additional requirement on  $\mathfrak{g}$ : namely, the Lie algebra  $\tilde{\mathfrak{g}}$  which has  $\mathfrak{g}$  as the underlying  $\mathbb{Z}_p$ -module and has the Lie bracket defined by  $[x, y]_{\tilde{\mathfrak{g}}} = 1/p \cdot [x, y]_{\mathfrak{g}}$  must be **pro-nilpotent** (equivalently,  $\tilde{\mathfrak{g}}/(p \cdot \tilde{\mathfrak{g}})$  must be a nilpotent Lie algebra over  $\mathbb{F}_p$ ). Thus, for example, Howe's result does not apply to groups such as the kernel of the reduction modulo  $p$  homomorphism  $GL_n(\mathbb{Z}_p) \rightarrow GL_n(\mathbb{F}_p)$  for any  $p \geq 3$ , whereas our results do apply to them.

The problem with the classical approach is that not every polarization of  $\mathfrak{g}$  corresponds to a subgroup of  $G$ , and Howe imposed his assumption precisely to deal with it. However, Andrei Jaikin-Zapirain showed in [JZ06] that Howe's assumption can be removed by proving the existence of polarizations satisfying some stronger conditions which allow the classical method to be used. His

Theorem 2.9 is equivalent to our Theorem 2.6. Moreover, he also obtained a result in the case  $p = 2$  (Theorem 2.12 in op. cit.) which is stronger than our Theorem 2.8. On the other hand, our result is already sufficient for some applications, as we demonstrate in the next section.

### 3. A $p$ -adic analogue of Brown's theorem

3.1. THE SETUP. We warn the reader that our notation here will differ from that of the first two sections. Namely, throughout the rest of the paper we let  $G$  be a  **$p$ -adic nilpotent Lie group**, and  $\mathfrak{g}$  its Lie algebra, which is a finite dimensional nilpotent Lie algebra over  $\mathbb{Q}_p$ . For our purposes one does not need to know the general definition of a  $p$ -adic Lie group; it suffices to think of  $G$  as the underlying topological space of  $\mathfrak{g}$  (where the topology on  $\mathfrak{g}$  is induced by the standard topology on  $\mathbb{Q}_p$ ) equipped with the operation given by the Campbell–Hausdorff series  $CH(x, y)$ . (Here there is no question of  $CH$  being well-defined or convergent, because  $\mathfrak{g}$  is a Lie algebra over a field of characteristic zero, and is nilpotent.) Thus  $G$  is a locally compact totally disconnected topological group. Recall also that a choice of a non-trivial continuous additive character  $\psi : \mathbb{Q}_p \rightarrow \mathbb{C}^\times$  allows one to identify  $\mathfrak{g}^*$  with  $\text{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, \mathbb{Q}_p)$ . The set of coadjoint orbits  $\mathfrak{g}^*/G$  is equipped with the quotient of the natural topology of  $\mathfrak{g}^*$ .

A complex representation  $(\pi, V)$  of  $G$ , where  $V$  is a vector space over  $\mathbb{C}$  and  $\pi : G \rightarrow GL(V)$  is a homomorphism, is said to be **algebraic** (or **smooth**) if for each  $v \in V$  the stabilizer  $G^v = \{g \in G : \pi(g)v = v\}$  is an open subgroup of  $G$ . Note that  $V$  need not have finite dimension in this definition, and, in fact, most irreducible algebraic representations of  $G$  are infinite-dimensional. The isomorphism class of  $(\pi, V)$  will be denoted by  $[(\pi, V)]$ , and we write  $\widehat{G}$  for the set of isomorphism classes of irreducible algebraic representations of  $G$ . It is equipped with the Fell topology, whose definition is recalled in §3.2 below. Calvin Moore proved [Mo65] that there is a natural bijection between  $\mathfrak{g}^*/G$  and  $\widehat{G}$  (see the introduction). The main result of this section is

**THEOREM 3.1:** *The orbit method bijection  $\mathfrak{g}^*/G \rightarrow \widehat{G}$  is a homeomorphism.*

Note that the continuity of this bijection is not difficult to check using an argument similar to the one for real Lie groups (but see also §3.5). On the other hand, it is rather non-trivial to prove that the bijection is open, and

our argument is based on a fact (Lemma 3.2) which does not have an obvious analogue over  $\mathbb{R}$ .

**3.2. FELL TOPOLOGY.** We recall the definition given in [GK92]. For an irreducible algebraic representation  $(\pi, V)$  of  $G$ , choose  $n \in \mathbb{N}$ , vectors  $v_1, \dots, v_n \in V$ , linear functionals  $\xi_1, \dots, \xi_n \in V^*$ , a compact set  $B \subset G$ , and a real number  $\epsilon > 0$ , and define

$$\mathcal{U}(\pi, V, B, v_j, \xi_j, \epsilon) \subseteq \widehat{G}$$

to be the set of isomorphism classes  $[(W, \rho)] \in \widehat{G}$  such that there exist  $w_1, \dots, w_n \in W$  and  $\eta_1, \dots, \eta_n \in W^*$  with the property

$$\left| \langle \xi_i, \pi(g)v_i \rangle - \langle \eta_i, \rho(g)w_i \rangle \right| < \epsilon \quad \text{for all } g \in B, 1 \leq i \leq n.$$

Sets of the form  $\mathcal{U}(\pi, V, B, v_j, \xi_j, \epsilon)$  are defined to be a basis of neighbourhoods of the point  $[(\pi, V)] \in \widehat{G}$ , which uniquely determines a topology on  $\widehat{G}$ , called the **Fell topology**. To understand it, we begin with the following

**LEMMA 3.2:** *If  $\mathfrak{g}$  is a finite dimensional nilpotent Lie algebra over  $\mathbb{Q}_p$ , then  $\mathfrak{g}$  can be written as the union of an increasing sequence of open uniform Lie subalgebras:*

$$(3.1) \quad \mathfrak{k}_1 \subseteq \mathfrak{k}_2 \subseteq \mathfrak{k}_3 \subseteq \dots \subseteq \mathfrak{g}, \quad \mathfrak{g} = \bigcup_{j \geq 1} \mathfrak{k}_j.$$

*Proof.* Let  $x_1, \dots, x_N$  be a basis of  $\mathfrak{g}$  over  $\mathbb{Q}_p$ . For every  $j \in \mathbb{N}$ , consider the set of all elements of  $\mathfrak{g}$  of the form

$$[y_1, [y_2, [y_3, [\dots [y_{t-1}, y_t] \dots ]]]],$$

where  $t \geq 1$  is arbitrary and each  $y_i$  is of the form  $p^{-j}x_k$  for some  $1 \leq k \leq N$ . Since  $\mathfrak{g}$  is nilpotent, only finitely many of these iterated commutators are non-zero, and hence their  $\mathbb{Z}_p$ -span, call it  $\mathfrak{k}'_j$ , is a free  $\mathbb{Z}_p$ -submodule of  $\mathfrak{g}$  of finite rank. Moreover,  $\mathfrak{k}'_j$  is closed under the Lie bracket by definition. By construction,  $\mathfrak{k}'_j \subseteq \mathfrak{k}'_{j+1}$  for all  $j \geq 1$ , and  $\mathfrak{g} = \mathfrak{k}'_1 \cup \mathfrak{k}'_2 \cup \dots$ . Let  $\mathfrak{k}_j = p \cdot \mathfrak{k}'_j$  (respectively,  $\mathfrak{k}_j = 4 \cdot \mathfrak{k}'_j$  if  $p = 2$ ); this is clearly a uniform Lie algebra, and since  $\mathfrak{g}$  is a vector space over  $\mathbb{Q}_p$ , it follows that (3.1) holds. Finally, each  $\mathfrak{k}_j$  is open because  $p^{1-j}x_m \in \mathfrak{k}_j$  for all  $1 \leq m \leq N$ . ■

3.3. SOME NOTATION. An obvious consequence of Lemma 3.2 is that one obtains the same topology on  $\widehat{G}$  by restricting the compact set  $B$  in the definition of the Fell topology to be an arbitrary open uniform subgroup  $K$  of  $G$ . Now either Theorem 2.6 or Theorem 2.8 applies to  $K$ . In order to make the argument below independent of  $p$ , let us define  $\alpha = 2$  if  $p = 2$  and  $\alpha = 1$  if  $p \geq 3$ . Put  $\mathfrak{k} = \text{Lie}(K) \subset \mathfrak{g}$ , and let  $\Omega_0 \subset (\alpha\mathfrak{k})^*$  be a  $K$ -orbit. If  $p = 2$ , then a finite subset  $\widehat{K}_{\Omega_0} \subset \widehat{K}$  was defined in Theorem 2.8. For  $p \geq 3$ , we let  $\widehat{K}_{\Omega_0} \subset \widehat{K}$  to be the singleton subset consisting of the irreducible representation of  $K$  corresponding to  $\Omega_0 \subset \mathfrak{k}^*$ . In either of the two cases, we define  $e_{\Omega_0} \in \text{Fun}(K^\alpha)^K$  be the pull-back via  $\log : K^\alpha \rightarrow \alpha\mathfrak{k}$  of the inverse Fourier transform of the characteristic function of  $\Omega_0$ , and Theorems 2.6 and 2.8 imply that if  $\rho \in \widehat{K}_{\Omega_0}$  and  $\chi_\rho$  is its character, then  $\chi_\rho|_{K^\alpha}$  is a multiple  $e_{\Omega_0}$ .

As the last piece of notation, if  $\pi$  is any complex continuous representation of  $K$ , we will denote by  $\text{supp}(\pi) \subseteq \widehat{K}$  the collection of all irreducible constituents of  $\pi$ .

3.4. PROOF OF THE DIFFICULT PART OF THEOREM 3.1. Given  $f \in \mathfrak{g}^*$ , we will prove that the orbit method bijection  $\mathfrak{g}^*/G \xrightarrow{\cong} \widehat{G}$  is open at the point  $\Omega_f \in \mathfrak{g}^*/G$ , where  $\Omega_f$  denotes the  $G$ -orbit of  $f$ . Consider an open neighbourhood of  $f$  in  $\mathfrak{g}^*$ . By shrinking it if necessary, we may assume, thanks to Lemma 3.2, that it is of the form

$$\mathcal{V}(f, K) = \left\{ f' \in \mathfrak{g}^* : f'|_{\alpha\mathfrak{k}} = f|_{\alpha\mathfrak{k}} \right\},$$

where  $K \subset G$  is an open uniform subgroup with Lie algebra  $\mathfrak{k}$  and  $\alpha$  is as in §3.3. We assume from now on that  $K$  is fixed. It suffices to check that the image of  $\mathcal{V}(f, K)$  under the orbit method map  $\mathfrak{g}^* \rightarrow \widehat{G}$  contains an open neighbourhood of  $[(\pi_f, V_f)]$  with respect to the Fell topology, where  $(\pi_f, V_f)$  denotes an irreducible algebraic representation of  $G$  corresponding to  $\Omega_f$ . The proof rests on the following

PROPOSITION 3.3: *Let  $\Omega \subset \mathfrak{g}^*$  be a  $G$ -orbit, let  $\Omega_0 \subset (\alpha\mathfrak{k})^*$  be a  $K$ -orbit, and let  $\pi$  denote the irreducible algebraic representation of  $G$  corresponding to  $\Omega$ . Then  $\text{supp}(\pi|_K) \cap \widehat{K}_{\Omega_0} \neq \emptyset$  if and only if  $\Omega_0$  is contained in the image of  $\Omega$  under the restriction map  $\text{res} : \mathfrak{g}^* \rightarrow (\alpha\mathfrak{k})^*$ .*

*Proof.* We use [GK92], §1.2. Recall that the character  $c(\pi)$  of  $\pi$  is not a function on  $G$ , but rather a distribution, defined by

$$(3.2) \quad \langle c(\pi), t \rangle = \text{tr} \left[ \int_G \overline{t(g)} \pi(g) dg \right], \quad t \in \widehat{C_0^\infty}(G),$$

where  $C_0^\infty(G)$  is the space of locally constant functions  $G \rightarrow \mathbb{C}$  with compact support, and  $dg$  denotes a fixed Haar measure on  $G$ . (The complex conjugation appears in the formula above for consistency with our orbit method for uniform pro- $p$ -groups.) Moreover, Kirillov's character formula in this context implies that  $\exp^*(c(\pi))$  is the inverse Fourier transform of a suitably normalized  $G$ -invariant measure on  $\mathfrak{g}^*$  supported on  $\Omega$ . Let  $e_0 = e_{\Omega_0} \in \text{Fun}(K^\alpha)^K \subset C_0^\infty(G)$  with the notation of §3.3. By definition,  $\text{supp}(\pi|_K) \cap \widehat{K}_{\Omega_0} \neq \emptyset$  if and only if  $e_0$  acts non-trivially on  $\pi|_K$ , which, in turn, is equivalent to  $\langle c(\pi), e_0 \rangle \neq 0$  because  $e_0$  is an idempotent in  $\text{Fun}(K)$ . Since the Fourier transform is an isometry, we see that  $\langle c(\pi), e_0 \rangle \neq 0$  if and only if  $\text{res}^{-1}(\Omega_0) \cap \Omega \neq \emptyset$ , which proves the proposition (since  $\text{res}(\Omega)$  is obviously  $K$ -stable). ■

Using the notation preceding the statement of the proposition, let  $f_0 = f|_{\alpha\mathfrak{k}}$ , let  $\Omega_0 \subset (\alpha\mathfrak{k})^*$  be the  $K$ -orbit of  $f_0$ , and let  $e_0 = e_{\Omega_0}$ . Then  $\Omega_0 \subseteq \text{res}(\Omega)$ , so by (the proof of) Proposition 3.3, we have  $\pi_f(e_0) \neq 0$ . In particular, there exist  $v \in V_f$  and  $\xi \in V_f^*$  such that  $\langle \xi, \pi_f(e_0)v \rangle = 1$ . Define  $\epsilon = \left( \int_K |e_0(k)| d\mu_K(k) \right)^{-1}$ , where  $\mu_K$  is the standard Haar measure on  $K$  of total mass 1. It is clear that the following result implies that the orbit method bijection  $\mathfrak{g}^*/G \rightarrow \widehat{G}$  is open.

**PROPOSITION 3.4:** *The open neighbourhood  $\mathcal{U}(\pi_f, V_f, K, v, \xi, \epsilon)$  of  $[(\pi_f, V_f)]$  in  $\widehat{G}$  is contained in the image of  $\mathcal{V}(f, K)$  under the map  $\mathfrak{g}^* \rightarrow \widehat{G}$ .*

*Proof.* Suppose that  $[(\rho, W)] \in \mathcal{U}(\pi_f, V_f, K, v, \xi, \epsilon)$ . By definition, there exist  $w \in W$  and  $\eta \in W^*$  such that  $|\langle \eta, \rho(g)w \rangle - \langle \xi, \pi_f(g)v \rangle| < \epsilon$  for all  $g \in K$ . Multiplying by the function  $|e_0(g)|$ , integrating with respect to  $\mu_K$  and using the definition of  $\epsilon$ , we obtain  $|\langle \eta, \rho(e_0)w \rangle - \langle \xi, \pi_f(e_0)v \rangle| < 1$ . By construction, this forces  $\langle \eta, \rho(e_0)w \rangle \neq 0$ , whence  $\rho(e_0) \neq 0$ , i.e.,  $\text{supp}(\rho|_K) \cap \widehat{K}_{\Omega_0} \neq \emptyset$ . By Proposition 3.3, if  $\Omega' \subset \mathfrak{g}^*$  is the  $G$ -orbit corresponding to  $\rho$ , then  $\text{res}(\Omega')$  contains  $f_0$ , i.e.,  $\Omega'$  meets  $\mathcal{V}(f, K)$ . ■

**3.5. PROOF OF CONTINUITY (SKETCH).** We conclude by briefly explaining how Lemma 3.2 can be used to prove that the orbit method bijection  $\mathfrak{g}^*/G \rightarrow \widehat{G}$  is continuous. We use the notation of §3.2. Fix  $(\pi, V) \in \widehat{G}$ , let  $\Omega \subset \mathfrak{g}^*$  be

the corresponding  $G$ -orbit, and consider a “standard” open neighbourhood of  $[(\pi, V)]$  of the form  $\mathcal{U} = \mathcal{U}(\pi, V, K, v_j, \xi_j, \epsilon)$ , where  $K \subset G$  is a uniform pro- $p$ -subgroup. In view of Lemma 3.2 it suffices to show that the inverse image of  $\mathcal{U}$  in  $\mathfrak{g}^*/G$  contains a neighbourhood of  $\Omega$ . To this end, let  $W \subseteq V$  be a finite dimensional  $K$ -invariant subspace containing all the  $v_j$ 's, let  $\vartheta$  denote the (possibly reducible) representation of  $K$  afforded by  $W$ , and let  $\chi_1, \dots, \chi_r \in \text{Fun}(K)^K \subset C_0^\infty(G)$  denote the characters of the irreducible constituents of  $\vartheta$ . It is clear that if  $(\pi', V') \in \widehat{G}$  is such that  $\pi'|_K$  contains a  $K$ -subrepresentation isomorphic to  $\vartheta$ , then  $[(\pi', V')] \in \mathcal{U}$ .

Now fix a Haar measure  $dg$  on  $G$ . For every  $G$ -orbit  $\Omega' \subset \mathfrak{g}^*$ , let  $\pi_{\Omega'}$  be the corresponding representation of  $G$ , let  $c(\pi_{\Omega'})$  be the character of  $\pi_{\Omega'}$  defined by (3.2), and let  $\mu_{\Omega'}$  denote the  $G$ -invariant measure on  $\mathfrak{g}^*$  supported on  $\Omega'$  whose inverse Fourier transform is equal to  $\exp^*(c(\pi_{\Omega'}))$ . Furthermore, let  $\mathfrak{k} = \text{Lie}(K) \subset \mathfrak{g}$ , let  $\text{res} : \mathfrak{g}^* \rightarrow \mathfrak{k}^*$  denote the restriction map, and let  $\nu_i : \mathfrak{k}^* \rightarrow \mathbb{C}$  denote the Fourier transform of  $\exp^*(\chi_i) \in \text{Fun}(\mathfrak{k})^K$ . It is not hard to check that as  $\Omega' \in \mathfrak{g}^*/G$  varies, the conditions  $\langle \nu_i, \text{res}_* \mu_{\Omega'} \rangle \geq \langle \nu_i, \text{res}_* \mu_\Omega \rangle$ ,  $1 \leq i \leq r$ , define an open subset  $\mathcal{V} \subseteq \mathfrak{g}^*/G$ . If  $\Omega' \in \mathcal{V}$ , then applying the inverse Fourier transform and Kirillov's character formula for the group  $G$ , we see that the multiplicity of each  $\chi_i$  in  $\pi_{\Omega'}|_K$  is at least its multiplicity in  $\pi|_K$ , whence  $\vartheta$  is isomorphic to a subrepresentation of  $\pi_{\Omega'}|_K$ . By the previous paragraph,  $\mathcal{V}$  is contained in the inverse image of  $\mathcal{U}$  in  $\mathfrak{g}^*/G$ , completing the proof.

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